Using Taylor's Inequality.

Proposition 1.4. Taylor's Inequality

Let $f(x)$ be $(N+1)$-times differentiable on $x=a$. Let $T_N(x)$ be the N^{th} degree Taylor polynomial of $f(x)$ about the base point $x=a$ with remainder term $R_N(x)$. Let $[a_L, a_U]$ be an interval containing the base point $x=a$. Then, for all $x \in [a_L, a_U]$:

$$|R_N(x)| \leq \frac{|x-a|^{N+1}}{(N+1)!}(M)$$

for some $M \in \mathbb{R}$ such that $|f^{(n+1)}(x)| \leq M$ for all $x \in [a_L, a_U]$. That is, M is an upper bound on the $(n+1)^{\text{th}}$ derivative. When convenient, M is sometimes chosen using

$$M = \max\left\{\left|f^{(N+1)}(c)\right| : c \in [a_L, a_U]\right\}$$

to get a stricter bound on the remainder term.

① Calculate $\cos(1)$ using $T_N(x)$ of $f(x) = \cos(x)$ about $a=0$. Find N such that $\cos(1)$ to 5 decimal places.

Soln: To find a bound involving $\cos(1)$, consider the interval $[0,1]$.

Since $f^{(N+1)} = \pm\cos(x)$ or $f^{(N+1)} = \pm\sin(x)$, choose $M=1$.

To be accurate to N decimal places, we want $|E_{max}| < \frac{1}{2}10^{-N}$. Consider $x=1$.

For 5 decimal places: $|R_N(x)| \leq \frac{|1-0|^{N+1}}{(N+1)!}M = \frac{1}{(N+1)!}(1) < \frac{1}{2}(10^{-5})$;

To solve $\frac{1}{(N+1)!} < \frac{1}{2}(10^{-5})$, we can find N minimal such that $(N+1)! > 2(10^5) = 200\,000$;
We apply brute force and use the fact that as N increases, $(N+1)!$ increases.

For $N=7$: $8! = 40\,320 \not> 200\,000$; For $N=8$: $9! = 362\,880 > 200\,000$; So, $\boxed{N=8 \text{ works.}}$

For 8 decimal places: Find N minimal such that $(N+1)! > 2(10^8)$;

For $N=10$: $11! = 39\,916\,800 = 3.99\ldots \times 10^7 \not> 2(10^8)$;
For $N=11$: $12! = 479\,001\,600 = 4.79\ldots \times 10^8 > 2\times 10^8$; $\boxed{\text{So, } N=11 \text{ works.}}$

② Let $T_5(x)$ be the 5^{th} deg Taylor poly of $f(x) = \cos(x)$ about $a=0$;
Find all values of $x \in [-1,1]$ such that $|R_5(x)| < 0.00214$;

Recall that $f^{(n)}(x) = \begin{cases} \cos(x) & \text{if } n = 4k \\ -\sin(x) & \text{if } n = 4k+1 \\ -\cos(x) & \text{if } n = 4k+2 \\ \sin(x) & \text{if } n = 4k+3 \end{cases}$ for some $k \in \mathbb{Z}$.

Then, $f^{(6)}(x) = -\cos(x)$; By properties of $\cos(x)$: $\max\{|\cos(x)| : x \in [-1,1]\} = \cos(1) \approx 0.540 < 0.541$

By Taylor's inequality with $M = 0.541$: $|R_5(x)| \leq \frac{|x-0|^6}{(5+1)!}M = \frac{|x|^6}{6!}(0.541) < 0.00214$;

By symmetry of x^6, we can assume $x \geq 0$.

$\frac{x^6}{6!}(0.541) < 0.00214$; $x^6 < 2.99767\ldots$; $(x^6)^{\frac{1}{6}} = x < (2.99767)^{\frac{1}{6}} = 1.20078\ldots$

since $f(x) = x^{\frac{1}{6}}$ is increasing.

Therefore, all $x \in (-1.20, 1.20)$ yields $|R_5(x)| < 0.00214$.

$\therefore$ All values within $\boxed{x \in [-1,1]}$ satisfy $|R_5(x)| < 0.00214$;